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The Lotka–Volterra equation over a finite ring $\mathbb{Z}/p^N\mathbb{Z}$

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Abstract

The discrete Lotka–Volterra equation over p-adic space was constructed since p-adic space is a prototype of spaces with non-Archimedean valuations and the space given by taking the ultra-discrete limit studied in soliton theory should be regarded as a space with the non-Archimedean valuations given in my previous paper (Matsutani S 2001 *Int. J. Math. Math. Sci.*). In this paper, using the natural projection from a p-adic integer to a ring $\mathbb{Z}/p^n\mathbb{Z}$, a soliton equation is defined over the ring. Numerical computations show that it behaves regularly.

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1. Introduction

According to [LL], studies on the ultrametric space which is characterized by the strong triangle axiom

$$d(x, z) \leqslant \max[d(x, y), d(y, z)] \tag{1}$$

or for the one-dimensional additive space case

$$|x - z|_{\mathrm{ult}} \leqslant \max[|x|_{\mathrm{ult}}, |z|_{\mathrm{ult}}] \tag{2}$$

have been going on for 10–15 years in the fields of general topology, computer language, rings of meromorphic function and so on. This space is called non-Archimedean space in English and German literature, ultrametric space in French literature and isosceles space in Russian literature [LL]. This space first appeared in number theory as a p-adic (Hensel) integer, but nowadays it is known that this space is natural even for the fields outside number theory.

In fact, the space obtained by the ultra-discrete limit which has been studied in soliton theory [TS, TTMS] holds relation (1) as shown in [M]. (If one recognizes that soliton theory is roughly a theory of functions over a compact Riemann surface, the ultrametric is built in soliton theory since the functions are also governed by the non-Archimedean valuation [I]. In fact, Sato theory is based on a valuation theory [SN].)

In the studies of the ultrametric space, it is natural to feel that the theory over a field with characteristics $q = 0$ is too restrictive because p-adic space is a prototype of the ultrametric space and the p-adic field has non-vanishing characteristics.

Actually as the ultra-discrete equations [TS, TTMS] are given as difference–difference equations, difference–difference equations are in general given by algebraic relations. We should deal with the equations from the viewpoint of the algebra and there non-vanishing characteristics is canonical. Accordingly we wish to formulate the difference–difference nonlinear equation over a more general field with non-vanishing characteristics or its related ring. In fact, there have been several attempts to formulate the soliton theory over finite fields [N, NM]. The purpose of this study (which includes my previous paper [M]) is to extend the soliton theory over $\mathbb{R}$ to p-adic number space in order to consider the meaning of the ultra-discrete limit. (As the p-adic valuation is a prototype of the non-Archimedean valuation and the ultra-discrete system is natural in soliton theory, it is very natural to investigate soliton equations in the p-adic space.) The extension of soliton theory to p-adic space was performed by Ichikawa for continuous soliton theory of KP-hierarchy [Ic1, Ic2, Ic3]. In this study, we restrict ourselves to only consider the difference–difference equations. Then as mentioned in [M], we can also define the p-adic difference–difference Lotka–Volterra equation and show that its p-adic valuation version has the same structure as the ultra-discrete difference–difference Lotka–Volterra equation.

In this paper, we will investigate the p-adic system more concretely. Since the p-adic integer is canonically connected with a finite ring $\mathbb{Z}/p^N\mathbb{Z}$, we will consider the Lotka–Volterra equation over the finite ring $\mathbb{Z}/p^N\mathbb{Z}$. Due to its finiteness, we can give explicit solutions. Numerical computations show that the Lotka–Volterra equation over the finite ring $\mathbb{Z}/p^n\mathbb{Z}$ behaves regularly.

In this paper, we denote the set of integers, rational numbers and real numbers by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ respectively.

2. The ultra-discrete limit as a valuation

This section briefly reviews the previous report [M] in order to connect the ultra-discrete system with the ultrametric system. Let $\mathcal{A}'_{[\beta]}$ be a set of non-negative real valued functions over $\mathbb{R}_{>0}$ where $\mathbb{R}_{>0}$ is a set of positive real numbers. We will denote the coordinate in $\mathbb{R}_{>0}$ by β and fix the notation. Let us define a correspondence $\mathrm{ord}_\beta : \mathcal{A}'_{[\beta]} \cup \{0\} \to \mathbb{R} + \infty$. We set $\mathrm{ord}_\beta(0) = \infty$ for $u \equiv 0$ and for $u \in \mathcal{A}'_{[\beta]}$, if there exists

$$\mathrm{ord}_\beta(u) := -\lim_{\beta\to+\infty} \frac{1}{\beta} \log(u(\beta)). \tag{3}$$

We call this value the ultra-discrete of u [TTMS].

Let us choose a subset $\mathcal{A}_{[\beta]}$ of $\mathcal{A}'_{[\beta]}$, $\mathcal{A}_{[\beta]} := \{u \in \mathcal{A}'_{[\beta]} \mid \mathrm{ord}_\beta(u)$ is well defined, $\mathrm{ord}_\beta(u) < \infty\}$. Further we identify the set $\{u \in \mathcal{A}'_{[\beta]} \mid \mathrm{ord}_\beta(u) = \infty\}$ with $\{0\}$. The ultra-discrete ord_β is a non-Archimedean valuation of $\mathcal{A}_{[\beta]} \cup \{0\}$ [I] since it possesses the properties (I_β):

Proposition I_β. *For $u, v \in \mathcal{A}_{[\beta]} \cup \{0\}$,*

(1) $\mathrm{ord}_\beta(uv) = \mathrm{ord}_\beta(u) + \mathrm{ord}_\beta(v)$.

(2) $\mathrm{ord}_\beta(u + v) = \min(\mathrm{ord}_\beta(u), \mathrm{ord}_\beta(v))$.

Let us now give the difference–difference Lotka–Volterra equation for $\{c_n^m \in \mathbb{R}_{\geqslant 0} \mid (n, m) \in \Omega \times \mathbb{Z}\}$ [HT],

$$\frac{c_n^{m+1}}{c_n^m} = \frac{1 + \delta c_{n-1}^m}{1 + \delta c_{n+1}^{m+1}}. \tag{4}$$

Here Ω is a subset of $\mathbb{Z}$, δ is a small parameter ($|\delta| < 1$) forming a connection between the discrete system and a continuum system, n is an index of a subset of the integer $\mathbb{Z}$ and m is a time step.

By introducing new variables $f_n^m := -\mathrm{ord}_\beta(c_n^m)$ and $d := -\mathrm{ord}_\beta(\delta)$ [T], we have an ultra-discrete version of the difference–difference Lotka–Volterra equation (5) for $c_n^m \in \mathcal{A}_{[\beta]}$ and $\delta \in \mathcal{A}_{[\beta]}$ [TS, T, TTMS],

$$f_n^{m+1} - f_n^m = \mathrm{ord}_\beta(1 + \delta c_{n-1}^m) - \mathrm{ord}_\beta(1 + \delta c_{n+1}^{m+1}). \tag{5}$$

We emphasize that (5) is considered as a valuation version of the difference–difference soliton equation (4).

Now in order to connect the ultra-discrete valuation and an ultrametric in the framework [LL], we introduce a real number $\overline{\beta} \gg 1$ and define a quantity for $x \in \mathcal{A}_{[\beta]}$ as

$$|x|_\beta := \left(\mathrm{e}^{-\overline{\beta}}\right)^{\mathrm{ord}_\beta(x)}. \tag{6}$$

This is an ultrametric because it satisfies the following proposition.

Proposition II$_\beta$. *For u, $v \in \mathcal{A}_{[\beta]} \cup \{0\}$, $|u|_\beta$ and $|v|_\beta$ the following properties hold:*

(1) $|u|_\beta$ depends upon $\overline{\beta}$.
(2) If $|v|_\beta = 0$, $v = 0$.
(3) $|v|_\beta \geqslant 0$.
(4) $|vu|_\beta = |v|_\beta |u|_\beta$.
(5) $|u + v|_\beta = \max(|u|_\beta, |v|_\beta) \leqslant |u|_\beta + |v|_\beta$.

Here we will remark on this ultrametric as follows [M].

(1) If we define the distance $d(x, y)$ between points x, $y \in \mathcal{A}_{[\beta]} \cup \{0\}$ by $d(x, y) := |\,|x - y|\,|_\beta$, the fifth property in II$_\beta$ satisfies (2), since the absolute value $|x - y|$ belongs to $\mathcal{A}_{[\beta]} \cup \{0\}$. This metric induces a very weak topology.
(2) Since $x \in \mathcal{A}_{[\beta]}$ has a finite value at $\beta \to \infty$, we have the relation

$$|x|_\beta|_{\overline{\beta}\sim\infty} \sim \exp(-\overline{\beta}(-(\log x)/\beta))|_{\overline{\beta}\sim\beta\sim\infty} = |x|^{\overline{\beta}/\beta}|_{\overline{\beta}\sim\beta\sim\infty}. \tag{7}$$

It may be considered that $|x|_\beta$ is identified with $|x|$ synchronizing $\overline{\beta}$ and β. $|x|_\beta$ is consistent with the natural absolute value metric $|\cdot|$ in $\mathbb{R}$.
(3) In this metric, we have the relation

$$\left|\sum_m x_m\right|_\beta = \mathrm{e}^{-\overline{\beta}\min(\mathrm{ord}_\beta(x_m))}. \tag{8}$$

This relation appears in the partition function at $\overline{\beta} \sim \beta = 1/T$, $T \to 0$ and in the semi-classical path integral $\overline{\beta} \sim \beta = 1/\hbar$, $\hbar \to 0$ [D, FH]. It means that *the classical regime appears as a non-Archimedean valuation, which is an algebraic manipulation*. For example, a problem with a minimal principle might be regarded as a valuation of a quantum problem.

These remarks show that the ultrametric obtained from the ultra-discrete is a very natural object from physical and mathematical viewpoints.

3. The p-adic difference–difference Lotka–Volterra equation and its applications

In the ultrametric space theory, the p-adic space is a prototype. Hereafter we will deal with a p-adic space. In this section, we define a p-adic difference–difference Lotka–Volterra equation and investigate it.

First let us introduce the p-adic field $\mathbb{Q}_p$ for a prime number p [C, I, L, RTV, VVZ]. For a rational number $u \in \mathbb{Q}$ which is given by $u = \frac{v}{w} p^m$ (v and w are coprime to the prime number p and m is an integer), we define a symbol $[[\cdot]]_p$ by $[[u]]_p := p^m$. Here let us define the p-adic valuation of u as a map from $\mathbb{Q}$ to a set of integers $\mathbb{Z}$,

$$\mathrm{ord}_p(u) := \log_p [[u]]_p \quad \text{for } u \neq 0 \qquad \text{and} \qquad \mathrm{ord}_p(u) := \infty \quad \text{for } u = 0. \tag{9}$$

This valuation has the following (I_p) properties, which is similar to I_β of ord_β:

Proposition $\mathbf{I}_p$. *For $u, v \in \mathbb{Q}$,*

(1) $\mathrm{ord}_p(uv) = \mathrm{ord}_p(u) + \mathrm{ord}_p(v)$.
(2) $\mathrm{ord}_p(u+v) \geqslant \min(\mathrm{ord}_p(u), \mathrm{ord}_p(v))$.
If $\mathrm{ord}_p(u) \neq \mathrm{ord}_p(v)$, $\mathrm{ord}_p(u+v) = \min(\mathrm{ord}_p(u), \mathrm{ord}_p(v))$.

This property (I_p-1) means that ord_p is a homomorphism from the multiplicative group $\mathbb{Q}^\times$ of $\mathbb{Q}$ to the additive group $\mathbb{Z}$. It implies that (I_β-1) should also be interpreted as a similar map. The p-adic metric is given by $|v|_p = p^{-\mathrm{ord}_p(v)}$, which has the properties (II_p);

Proposition $\mathbf{II}_p$. *For $u, v \in \mathbb{Q}$,*

(1) if $|v|_p = 0$, $v = 0$.
(2) $|v|_p \geqslant 0$.
(3) $|vu|_p = |v|_p |u|_p$.
(4) $|u+v|_p \leqslant \max(|u|_p, |v|_p) \leqslant |u|_p + |v|_p$.

The p-adic field $\mathbb{Q}_p$ is given as a completion of $\mathbb{Q}$ with respect to this metric so that properties (I_p) and (II_p) survive for $\mathbb{Q}_p$.

It should be noted that the properties (I_p) and (II_p) are essentially the same as those of (I_β) and (II_β). As a property of the p-adic metric, the convergence condition of the series $\sum_m x_m$ is identified with the vanishing condition of sequence $|x_m|_p \to 0$ for $m \to \infty$ due to the relationship [C, L, VVZ],

$$\left| \sum_m x_m \right|_p = \max |x_m|_p. \tag{10}$$

This property is also related to (8).

Furthermore, we introduce the integer part of $\mathbb{Q}_p$, $\mathbb{Z}_p := \{u \in \mathbb{Q}_p \mid \mathrm{ord}_p(u) > 0\}$ so that we define a p-adic soliton equation.

As we finish the preparation, let us define the p-adic difference–difference Lotka–Volterra equation for a p-adic series $\{c_n^m \in \mathbb{Q}_p \mid (n, m) \in \Omega \times \mathbb{Z}\}$ ($p \neq 2$),

$$\frac{c_n^{m+1}}{c_n^m} = \frac{1 + \delta_p c_{n-1}^m}{1 + \delta_p c_{n+1}^{m+1}} \tag{11}$$

or

$$c_n^{m+1}(1 + \delta_p c_{n+1}^{m+1}) = c_n^m (1 + \delta_p c_{n-1}^m) \tag{12}$$

where $\delta_p \in p\mathbb{Z}_p - \{0\}$. Here we note that $\mathbb{Z}_p$ is a *local ring*, i.e. it has only prime ideals $\{0\}$ and $p\mathbb{Z}_p$ [L]. Hence for any $c, c' \in p\mathbb{Z}_p$ and $x, y \in \mathbb{Z}_p$, $|xc + yc'|_p < 1$, and for any $x \in \mathbb{Z}_p - \{0\}$

and $i, j \in \mathbb{Z}$ $(i > j > 0)$, $0 < |x\delta_p^i|_p < |x\delta_p^j|_p < 1$. As we show later these properties are essential in order that the p-adic soliton equation (12) is well defined. We proved that this equation has non-trivial solutions in [M] by showing the existence of the soliton solutions. The proof in [M] is also based on the property that $\mathbb{Z}_p$ is a local ring.

In this paper, we will give another proof. Since it is known that there is a natural projection from $\mathbb{Z}_p$ to $\mathbb{Z}/p^N\mathbb{Z}$ [L], equation (12) can be solved by modulo computations if $c_n^0 \in \mathbb{Z}_p$ for all $n \in \Omega$. Let us expand c_n^m in the p-adic space $\mathbb{Z}_p$,

$$c_n^m := \alpha_n^{m\,(0)} + \alpha_n^{m\,(1)} p + \alpha_n^{m\,(2)} p^2 + \cdots \qquad c_n^{m\,(N)} := \sum_{i=0}^{N-1} \alpha_n^{m\,(i)} p^i \tag{13}$$

where $\alpha_n^{m\,(i)} = 0, 1, \ldots, p-1$. Since p is a small parameter in $\mathbb{Z}_p$ and we have relation (10), this series converges if $\alpha_n^{m\,(n)} \to 0$ for $n \to \infty$. This expansion corresponds to the Taylor expansions, e.g. with respect to a small parameter δ, in ordinary metric space. Furthermore, it is noted that the truncated series $c_n^{m\,(N)}$ is defined over $\mathbb{Z}/p^N\mathbb{Z}$ and the projection ϕ (truncation or modulo computation) from $\mathbb{Z}_p$ to $\mathbb{Z}/p^N\mathbb{Z}$ preserves the operations of addition and multiplication; for $c, c' \in \mathbb{Z}_p$, $\phi(c + c') = \phi(c) + \phi(c')$ and $\phi(cc') = \phi(c)\phi(c')$. Hence, in other words, ϕ is a surjective ring homomorphism.

For simplicity, we assume $\delta_p \equiv p$. By comparing the coefficients of p^i $(i = 0, 1, 2, \ldots)$ in both sides of (12), we can determine the time revolution iteratively:

$$\begin{aligned}
\alpha_n^{m+1\,(0)} &= \alpha_n^{m\,(0)} \\
\alpha_n^{m+1\,(1)} &= \left(c_n^m(1 + pc_{n-1}^m) - c_n^{m+1\,(1)}\right)\Big/\, p \quad \text{modulo } p \\
\cdots \\
\alpha_n^{m+1\,(N')} &= \left(c_n^m(1 + pc_{n-1}^m) - c_n^{m+1\,(N')}(1 + p\, c_{n+1}^{m+1\,(N')})\right)\Big/\, p^{N'} \quad \text{modulo } p.
\end{aligned} \tag{14}$$

This comparison (14) means that we compute (12) in modulo $p^{N'+1}$ $(N' = 0, 1, 2, \ldots)$. The fact that these coefficients $\alpha_n^{m+1\,(i)}$ are determined from (14) is also due to the property that $\mathbb{Z}_p$ is a local ring.

Suppose that the initial state is given by $\{c_n^0 \equiv c_n^{0\,(N_0)}\}_{n\in\Omega}$ for a finite N_0 or $\alpha_n^{0\,(i)} \equiv 0$ for $n \in \Omega$, $i > N_0 + 1$. Then the above computations (14) determine all values of $\alpha_n^{m\,(i)}$ $i = 0, 1, 2, \ldots$ and $m > 0$. Then for a finite m, $\alpha_n^{m\,(i)}$ vanish for $n \in \Omega$, $i > N_1$, and sufficiently large N_1. It implies that (12) has non-trivial solutions in p-adic space.

Using the natural projection from $\mathbb{Z}_p$ to $\mathbb{Z}/p^N\mathbb{Z}$, we have more concrete solutions (14) of equation (12) modulo p^N when we fix N. We should note that since the projection is a ring homomorphism, equation (12) modulo $p^N\mathbb{Z}$ also has the same form. In other words, we can define a difference–difference Lotka–Volterra equation over $\mathbb{Z}/p^N\mathbb{Z}$. We give some examples of its solutions in tables 1–3 with a periodic boundary condition for n; $c_n^m \equiv c_{n+M}^m$ and $M = 5$, i.e. $\Omega = \mathbb{Z}/5\mathbb{Z}$. These are the $p = 3$ and $p = 5$ modulo p^3 $(N = 3)$ cases and $p = 7$ modulo p^2 $(N = 2)$ case. By letting $c^m := \{c_n^m | n \in \Omega\}$, we regard c^m as a wavefunction over Ω at the time m. Then equation (12) and these tables can be regarded as time developments of the wavefunctions c^m. Numerical computations show that they are also periodic on time m, $c^m = c^{m+p^{N-1}}$, and there appears no travelling wave. Since $\alpha_n^{m\,(0)}$ $(n \in \Omega)$ are conserved quantities for the time development, i.e. $\alpha_n^{m+1\,(0)} = \alpha_n^{m\,(0)}$ for all n, the invariance might prohibit travelling waves. Further we note that there are several symmetries; in the tables c_4^m oscillate with shorter periods p^{N-2}, $\{c_n^m\}$ have point symmetry centrizing at $(n = 3/2, m = p^{N-1}/2)$, and so on. We consider that these are non-trivial Lotka–Volterra solutions over rings $\mathbb{Z}/p^N\mathbb{Z}$. It is also noted that (12) over the finite field $\mathbb{F}_p \equiv \mathbb{Z}/p\mathbb{Z}$ gives only trivial solutions since

Table 1. Modulo 3^3, $\delta_p = 3$.

	n					
m	0	1	2	3	4	5
0	1	2	2	1	1	1
1	7	23	26	22	10	7
2	13	8	14	16	10	13
3	19	11	20	10	1	19
4	25	5	17	4	10	25
5	4	17	5	25	10	4
6	10	20	11	19	1	10
7	16	14	8	13	10	16
8	22	26	23	7	10	22
9	1	2	2	1	1	1

Table 2. Modulo 5^3, $\delta_p = 5$.

	n					
m	0	1	2	3	4	5
0	1	2	2	1	1	1
1	96	117	37	106	26	96
2	16	7	97	36	101	16
3	11	47	57	41	101	11
4	81	112	42	121	26	81
5	101	77	52	26	1	101
6	71	67	87	6	26	71
7	116	82	22	61	101	116
8	111	122	107	66	101	111
9	56	62	92	21	26	56
10	76	27	102	51	1	76
11	46	17	12	31	26	46
12	91	32	72	86	101	91
13	86	72	32	91	101	86
14	31	12	17	46	26	31
15	51	102	27	76	1	51
16	21	92	62	56	26	21
17	66	107	122	111	101	66
18	61	22	82	116	101	61
19	6	87	67	71	26	6
20	26	52	77	101	1	26
21	121	42	112	81	26	121
22	41	57	47	11	101	41
23	36	97	7	16	101	36
24	106	37	117	96	26	106
25	1	2	2	1	1	1

$c_n^{m\,(1)} \equiv \alpha_n^{m\,(0)}$ is invariant for the time development. These facts mean that *we can define a soliton equation over rings beside finite fields* [N, NM].

Here we will mention the relation between the p-adic equation (12) and the ultra-discrete system following [M]. As the p-adic difference–difference Lotka–Volterra equation is well defined, let us consider the p-adic valuation of equation (11) even though the conserved

Table 3. Modulo 7^2, $\delta_p = 7$.

	n					
m	0	1	2	3	4	5
0	1	2	2	1	1	1
1	43	37	16	8	1	43
2	36	23	30	15	1	36
3	29	9	44	22	1	29
4	22	44	9	29	1	22
5	15	30	23	36	1	15
6	8	16	37	43	1	8
7	1	2	2	1	1	1

quantities $\alpha_n^{m\,(0)}$ might make the valuation trivial. By letting $f_n^m := -\mathrm{ord}_p(c_n^m)$ and $d_p := -\mathrm{ord}_p(\delta_p)$, we have

$$f_n^{m+1} - f_n^m = \mathrm{ord}_p(1 + \delta_p c_{n-1}^m) - \mathrm{ord}_p(1 + \delta_p c_{n+1}^{m+1}). \tag{15}$$

For $f_n^m \neq -d_p$, (15) becomes

$$f_n^{m+1} - f_n^m = \max(0, f_{n-1}^m + d_p) - \max(0, f_{n+1}^{m+1} + d_p). \tag{16}$$

We emphasize that (15) has the same form as the ultra-discrete difference–difference Lotka–Volterra equation (5) [M]. Both stand upon the same structure. Since the p-adic valuation is a natural object in the p-adic number and the family of rings $\{\mathbb{Z}/p^N\mathbb{Z}\}$, the ultra-discrete difference–difference system should also be studied from the point of view of valuation theory [M].

As we finish this section, we will give a comment on a relation to q-analysis. It is known that some of properties in the q analysis can be regarded as those in p-adic analysis by setting $q = 1/p$ [VVZ]. We have correspondence among p, q and e^β as [M],

$$\mathrm{e}^{-\beta} \Longleftrightarrow p\ (\beta \sim \infty) \qquad p \Longleftrightarrow 1/q \qquad q \Longleftrightarrow \mathrm{e}^{\beta}\ (\beta \sim 0). \tag{17}$$

4. Summary and discussion

We showed that the ultra-discrete limit should be regarded as a non-Archimedean valuation following my previous paper [M]. After we constructed the ultrametric related to the ultra-discrete limit in (6), we remarked upon its properties. As a consequence of the remarks, it is interpreted that the ultra-discrete limit is a very natural manipulation in $\mathcal{A}_\beta \cup \{0\}$.

Generally in studies of the ultrametric space [LL], the p-adic system is a prototype. Thus we considered the p-adic soliton equation following my previous paper [M]. In fact the structure of the p-adic valuation of the p-adic difference–difference equation is the same as that of the ultra-discrete difference–difference equation for the case of the Lotka–Volterra equation.

Furthermore, since the p-adic integer $\mathbb{Z}_p$ has a natural projection to a finite ring $\mathbb{Z}/p^N\mathbb{Z}$, we have studied the Lotka–Volterra equation over the finite ring. Due to the finiteness of the system, we can give concrete solutions for the equation. There remains an open problem of what is integrability in the sense of $\mathbb{Z}/p^N\mathbb{Z}$ but the numerical computations give regular results and several symmetries of the system; it is expected that our system is 'integrable' in a sense. In other words, the wavefunctions c^m are periodic in time and recover the original $c^{m=0}$ at $m = p^{N-1}$ after they are distorted. This means that for all steps m, they preserve the original data of $c^{m=0}$. Further they have a point symmetry and c_4^m have shorter periods in each table. The point symmetry implies that there exists a space-time reversal symmetry, i.e. for the exchange $(n, m) \to (3 - n, -m)$, the tables do not change modulo $(5\mathbb{Z}, p^{N-1}\mathbb{Z})$. It is expected that there is a group governing this system.

This construction of the p-adic equation can be easily extended to a more general p-adic system over an algebraic integer if the algebraic integer is a principal domain. As soliton theory in the finite field is closely related to code theory [N, NM], the soliton over $\mathbb{Z}/p^N\mathbb{Z}$ might also be applied to the information theory [K].

Finally we comment upon another open problem. The non-Archimedean valuation theory is associated with measure theory or non-standard statistics [LL] and renormalization theory [RTV]. On the other hand, soliton theory is connected with the statistical system and statistical mechanics [So]. Thus we are posed with the question of whether both non-standard statistics and soliton theory have a more direct relation.

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